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Letter to the Editor

Approximate eigensolutions of axially moving beams with small flexural stiffness

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1. Introduction

Belt-pulley drives are widely used to drive a variety of machinery, including serpentine belt systems, band saws, tape drives, etc. In cases where the transport speed of the belt is high, these systems have the gyroscopic characteristics of axially moving continua. The transverse vibration of these axially moving continua is typically modelled as either a travelling string or a travelling, tensioned, Euler–Bernoulli beam. Closed-form solutions for the natural frequencies and vibration modes are available for the string model [1,2]. For the axially moving beam model, due to the beam's dispersive property, only numerical solutions are available [3,4]. Because most axially moving media have small bending stiffness relative to their tension, they can be modelled as an axially moving beam with small dimensionless bending stiffness. The transition of modal properties from the known axially moving string case to the beam case is desirable from both practical and theoretical viewpoints. Finding closed-form approximate solutions of the eigenvalue problem for such transitional systems is the main objective of this study. Before highlighting the works in this field, we first review work related to the phase closure principle, which is one of the main tools used in this article.

The vibration of elastic structures can be described in terms of waves propagating and attenuating in structures. The phase closure principle [5] states that if the phase change for propagating (or evanescent) waves is an integer multiple of 2π as they return to their start point after travelling forward and back along a finite structure, then the frequency at which the waves travel is a natural frequency and the corresponding vibration mode is the superposition of the component waves. In the field of acoustics, solids, and fluids, wave propagation and attenuation in waveguides and wave reflection/transmission at a boundary point has been studied extensively. The phase closure principle links the knowledge of wave motion in these fields to computation of natural frequencies and modes of finite structures. Mead [6] applies the method to find the eigensolutions of stationary beams. Exact frequency equations are established that differ from the

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conventional ones but have identical roots. These frequency equations have clear physical interpretation and deepen understanding of the beam vibration modes. Mace [7] develops a numerical matrix method based on wave propagation, reflection, and transmission at a point support (or a change of cross-section or material property) to calculate the natural frequencies and modes for beams. Tan and his co-workers [8,9] extend this method to some complex beam structures, like those consisting of several different uniform segments.

For axially moving continua it is well known that the vibration modes can be viewed as the superposition of pairs of opposite-going propagating waves. The phase speeds in the opposite directions are different due to the convective effect of the medium's axial speed. Lengoc and McCallion [10] study the relation between wave propagation and natural frequency, but their work is limited to non-dispersive system like taut strings. Lee and Mote [11] investigate the energy transfer due to the interaction between the translating continua and its boundary supports. The phase closure principle is used to obtain the natural frequencies of an axially moving string. Chakraborty and Mallik [12] study the free vibration of a travelling beam simply supported at both ends. The frequency equation is derived based on the phase closure principle. This work applies only to beams with finite bending stiffness and zero tension; transition behavior from a taut string to a tensioned beam is not investigated.

More commonly, researchers investigate axially moving continua mathematically without consideration of the physical wave propagation. They study transition behaviors for moving beams with vanishing bending stiffness by using perturbation techniques directly on the differential equations. Because the main concern of this study is the modal properties, only those aspects of related works are reviewed here. Pellicano and Zirilli [13] study axially moving beams with simple supports at both ends. While not specifically addressed, their natural frequencies can be extracted from the results. These natural frequencies depend only on the displacement boundary condition of each end, suggesting that the remaining two beam boundary conditions do not affect the natural frequencies. Öz et al. [14] and Özkaya and Pakdemirli [15] examine the transition from axially moving string to beam for an axially accelerating material. By letting the accelerating terms vanish, the free vibration solutions for constant belt speed follow. In Ref. [14], multiple scales perturbation is applied to find the approximate natural frequencies. Problems are apparent because no boundary conditions are considered in the derivation, indicating that different boundary conditions yield the same natural frequencies. The problem considered in Ref. [15] is similar to that in Ref. [14] and similar techniques are used. The improvement is that the spatial boundary layer terms arising from small bending stiffness are considered. Two sets of boundary conditions are considered. The solutions in Ref. [15] incorrectly imply that the approximate natural frequencies for these two sets of boundary conditions are the same. Further, for clamped boundaries the zero speed solution fails to give the approximate solution given by O'Malley [16]. In contrast, the present analysis gives different natural frequencies for these two kinds of boundary conditions, and the zero speed results converge to the exact solution (simply supported) and that given by O'Malley (fixed–fixed), although the adopted methods differ. O'Malley's [16] work treats stationary beams with two clamped ends. When we extended this method to axially moving beams, the procedure became cumbersome and no explicit solutions were obtained.

In this study, a different perturbation method is developed to find closed-form, approximate eigensolutions of axially moving beams with small bending stiffness. Wave propagation considerations lead to an algebraic equation with a small dimensionless bending stiffness parameter. Taking advantage of the simplicity of the propagation and attenuation properties of the waves, which are determined by the roots of an algebraic equation, the phase closure principle is used to find the natural frequencies. The complex vibration modes are obtained naturally from the superposition of all component waves in the beam. Approximate eigensolutions for different boundary conditions are presented. The perturbation solutions are confirmed by comparison with numerically exact ones. For the special cases mentioned above where the exact or approximate solutions are available, the derived approximate solutions agree with them.

Instead of considering spatial and temporal variations for the governing partial differential equation (like in Refs. [13,15]), this approach focuses on perturbation of algebraic equations. No boundary layers or secular terms need to be considered explicitly in the derivation. Although the method is simple, no completeness of the solutions is sacrificed. For example, the evanescent wave components (if not zero) automatically generate boundary layer terms for those beams where a small bending stiffness creates edge effects at the boundaries. Unlike prior perturbations where assumed mode spatial expansions are only suited for certain boundary conditions, this method handles different boundary conditions with a consistent treatment.

2. Model equations

The dynamic equation for an axially moving beam is

$$mw_{tt} + 2mcw_{xt} - (P - mc^2)w_{xx} + EIw_{xxxx} = 0, \quad 0<x<L, \tag{1}$$

where m is the belt mass per unit length, $w(x,t)$ is the transverse displacement, c is the belt transport speed, P is the tension, and EI is the bending stiffness. The following non-dimensional variables are introduced:

$$\hat{x} = \frac{x}{L}, \quad \hat{w} = \frac{w}{L}, \quad \hat{t} = t\sqrt{\frac{P}{mL^2}}, \quad \varepsilon^2 = \frac{EI}{PL^2}, \quad \hat{v} = c\sqrt{\frac{m}{P}}. \tag{2}$$

Substitution of Eq. (2) into Eq. (1) leads to the dimensionless equation (after dropping the hat)

$$w_{tt} + 2vw_{xt} - (1 - v^2)w_{xx} + \varepsilon^2 w_{xxxx} = 0, \quad 0<x<1. \tag{3}$$

Assuming $w = e^{i(rx-\omega t)}$, where r is the wavenumber and ω is the wave propagation frequency, Eq. (3) yields

$$\varepsilon^2 r^4 + (1 - v^2)r^2 + 2v\omega r - \omega^2 = 0. \tag{4}$$

Note that the small parameter $\varepsilon \ll 1$ multiplies the highest power of r. The roots of such a polynomial equation have two possible forms [17].

In the first form, the roots of Eq. (4) are expressed using the straightforward expansion

$$r = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots \tag{5}$$

Substitution of Eq. (5) into Eq. (4) leads to the ε^0 order result

$$x_0 = \frac{\omega}{v \pm 1} \tag{6}$$

and the ε^1 order result

$$2(1-v^2)x_0 + 2v\omega = 0 \quad \text{or} \quad x_1 = 0. \tag{7}$$

The first equation in Eq. (7) is discarded because it contradicts Eq. (6), so $x_1 = 0$. The ε^2 order equation gives

$$x_2 = \frac{-x_0^4}{2v\omega + 2(1-v^2)x_0} = \begin{cases} -\dfrac{1}{2}\dfrac{\omega^3}{(v+1)^4} & \text{when } x_0 = \dfrac{\omega}{v+1}, \\ \dfrac{1}{2}\dfrac{\omega^3}{(1-v)^4} & \text{when } x_0 = \dfrac{-\omega}{1-v}. \end{cases} \tag{8}$$

Eqs. (5)–(8) provide two of the four roots of Eq. (4).

The remaining two roots are expressed as the singular expansion

$$r = \frac{y}{\varepsilon^\lambda} + x_0 + \cdots, \quad \lambda > 0. \tag{9}$$

Substitution of Eq. (9) into Eq. (4) gives

$$\varepsilon^2\left(\frac{y}{\varepsilon^\lambda} + x_0 + \cdots\right)^4 + (1-v^2)\left(\frac{y}{\varepsilon^\lambda} + x_0 + \cdots\right)^2 + 2v\omega\left(\frac{y}{\varepsilon^\lambda} + x_0 + \cdots\right) - \omega^2 = 0. \tag{10}$$

The dominant terms in Eq. (10) are $y^4/\varepsilon^{4\lambda-2}$ and $(1-v^2)y^2/\varepsilon^{2\lambda}$. Balancing these leads to

$$4\lambda - 2 = 2\lambda \quad \Rightarrow \quad \lambda = 1. \tag{11}$$

Eq. (10) then becomes

$$\varepsilon^{-2}[y^4 + (1-v^2)y^2] + \varepsilon^{-1}[4x_0y^3 + 2(1-v^2)x_0y + 2v\omega y] + \cdots = 0 \tag{12}$$

with the solutions

$$y = \pm \mathrm{i}\sqrt{1-v^2}, \tag{13}$$

$$2x_0[(1-v^2) - 2(1-v^2)] + 2v\omega = 0 \quad \Rightarrow \quad x_0 = \frac{v\omega}{1-v^2}. \tag{14}$$

In summary, the wave dispersion equation (4) has the four roots

$$r_1 = \frac{\omega}{1+v} - \varepsilon^2\frac{1}{2}\frac{\omega^3}{(1+v)^4} + O(\varepsilon^3), \qquad r_2 = -\frac{\omega}{1-v} + \varepsilon^2\frac{1}{2}\frac{\omega^3}{(1-v)^4} + O(\varepsilon^3),$$
$$r_3 = \frac{v\omega}{1-v^2} + \mathrm{i}\frac{\sqrt{1-v^2}}{\varepsilon} + O(\varepsilon), \qquad r_4 = \frac{v\omega}{1-v^2} - \mathrm{i}\frac{\sqrt{1-v^2}}{\varepsilon} + O(\varepsilon). \tag{15}$$

Table 1 compares the approximate and numerically exact roots for three cases. Consistent with Eq. (15), r_1 and r_2 are best approximated by perturbation. Physically, the real parts of the roots represent the phase change between two points a unit distance apart, and the imaginary parts represent the variation of the wave amplitudes for two points a unit distance apart. Specifically, r_1 represents the wave propagating in the positive direction, r_2 the wave propagating in the negative direction, r_3 the evanescent wave attenuating in the positive direction, and r_4 the evanescent wave attenuating in the negative direction. The beam motion is the superposition of the

Table 1
Comparison of approximate roots of Eq. (4) from Eq. (15) with numerically exact roots

Case	Results	r_1	r_2	r_3	r_4
$\varepsilon = 0.01$	Exact	0.7222	−6.4335	2.8556 + 60.2421i	2.8556 − 60.2421i
$v = 0.80$					
$\omega = 1.30$	Approx.	0.7222	−6.4313	2.8889 + 59.9999i	2.8889 − 59.9999i
$\varepsilon = 0.05$	Exact	0.6482	−1.1501	0.2510 + 19.2243i	0.2510 − 19.2243i
$v = 0.28$					
$\omega = 0.83$	Approx.	0.6482	−1.1501	0.2522 + 19.2000i	0.2522 − 19.2000i
$\varepsilon = 0.10$	Exact	1.2949	−1.7342	0.2197 + 10.0070i	0.2197 − 10.0070i
$v = 0.15$					
$\omega = 1.50$	Approx.	1.2947	−1.7324	0.2302 + 9.8869i	0.2302 − 9.8869i

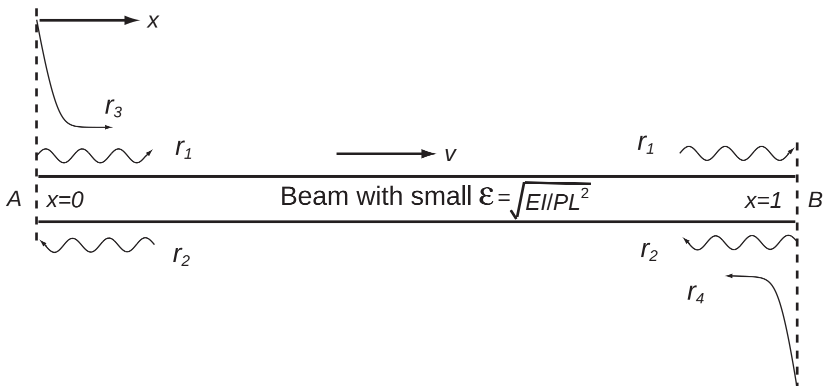

Fig. 1. Waves in a finite moving beam with small bending stiffness.

four components

$$w(x,t) = [c_1 e^{ir_1 x} + c_2 e^{ir_2 x} + c_3 e^{ir_3 x} + c_4 e^{ir_4 x}]e^{-i\omega t} \tag{16}$$

where c_1–c_4 are complex coefficients.

For small bending stiffness $\varepsilon \ll 1$, the imaginary parts of the evanescent waves r_3 and r_4 become very large. Consequently the r_3 component can exist only close to the boundary $x = 0$, and the r_4 component exists only close to the boundary $x = 1$ (Fig. 1). They can be viewed as part of the reflected waves as the propagating waves (r_1 and r_2) travel forward and back along the beam between the boundary points A ($x = 0$) and B ($x = 1$) (Fig. 1). The phase closure principle can now be applied to the propagating waves to find the eigensolutions for different boundary conditions.

3. Application of the phase closure principle

To apply the phase closure principle to the propagating waves (r_1 and r_2), one needs to find four different phase changes: first, the propagating wave leaves from boundary A and arrives at boundary B with a phase change $\mathrm{Re}\,(r_1) = r_1$ (because the span is normalized to unit length); second, it reflects at boundary B with a phase change $\phi(R_B)$; third, it travels from boundary B to boundary A with another phase change $-\mathrm{Re}(r_2) = -r_2$ (minus sign due to leftward propagation); finally, it reflects at boundary A with the fourth phase change $\phi(R_A)$ and returns to the start point boundary A. Mathematically, the phase closure principle requires

$$r_1 + \phi(R_B) - r_2 + \phi(R_A) = 2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots \tag{17}$$

Consider the case of a simply supported beam with boundary conditions

$$w(0,t) = w_{xx}(0,t) = 0, \qquad w(1,t) = w_{xx}(1,t) = 0 \tag{18}$$

At $x = 0$ (point A in Fig. 1), there is no r_4 evanescent component as noted above, and Eq. (16) becomes

$$w(x,t)|_A = [c_1 \mathrm{e}^{\mathrm{i}r_1x} + c_2 \mathrm{e}^{\mathrm{i}r_2x} + c_3 \mathrm{e}^{\mathrm{i}r_3x}]\mathrm{e}^{-\mathrm{i}\omega t} \tag{19}$$

Substitution into the boundary conditions at $x = 0$ yields

$$\begin{pmatrix} 1 & 1 \\ r_1^2 & r_3^2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_3 \end{pmatrix} = -\begin{pmatrix} 1 \\ r_2^2 \end{pmatrix} c_2 \quad \Rightarrow \quad \begin{pmatrix} c_1 \\ c_3 \end{pmatrix} = \frac{1}{r_3^2 - r_1^2}\begin{pmatrix} r_2^2 - r_3^2 \\ r_1^2 - r_2^2 \end{pmatrix} c_2. \tag{20}$$

For the two propagating waves, the relative phase due to the reflection at the left boundary is given by the phase of

$$R_A = \frac{c_1}{c_2} = -\frac{r_3^2 - r_2^2}{r_3^2 - r_1^2} = -\frac{\{-(1-v^2) + \varepsilon^2[v\omega/(1-v^2)]^2 - \varepsilon^2[\omega/(1-v)]^2\} + \mathrm{i}2\varepsilon v\omega/\sqrt{1-v^2}}{\{-(1-v^2) + \varepsilon^2[v\omega/(1-v^2)]^2 - \varepsilon^2[\omega/(1+v)]^2\} + \mathrm{i}2\varepsilon v\omega/\sqrt{1-v^2}}. \tag{21}$$

One can prove mathematically that the phase angle of R_A is

$$\phi(R_A) = \pi + O(\varepsilon^3) \tag{22}$$

as shown graphically in Fig. 2, where π is from the leading minus sign preceding $(r_3^2 - r_2^2)/(r_3^2 - r_1^2)$ in Eq. (21).

At $x = 1$ (point B), there is no r_3 evanescent component in Eq. (16). In seeking the relative phase between propagating waves at B, it is notationally convenient to introduce $\xi = x - 1$ and express the coefficients in Eq. (16) using $b_k = c_k \mathrm{e}^{\mathrm{i}r_k}$. This gives

$$w(x,t)|_B = [b_1 \mathrm{e}^{\mathrm{i}r_1\xi} + b_2 \mathrm{e}^{\mathrm{i}r_2\xi} + b_4 \mathrm{e}^{\mathrm{i}r_3\xi}]\mathrm{e}^{-\mathrm{i}\omega t}. \tag{23}$$

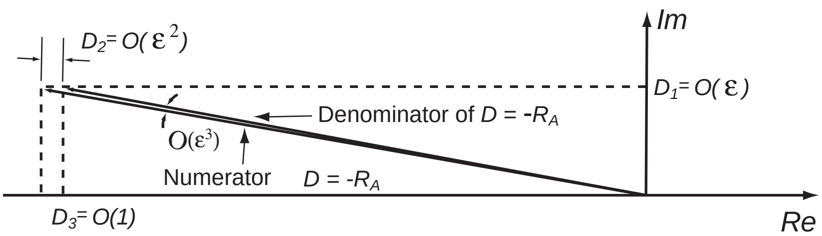

Fig. 2. Phase angle of $D = -R_A = (r_3^2 - r_2^2)/(r_3^2 - r_1^2)$ from Eq. (21); $D_1 = 2\varepsilon v\omega/\sqrt{1-v^2}$, $D_2 = \varepsilon^2\{[\omega/(1-v)]^2 - [\omega/(1+v)]^2\}$, $D_3 = -(1-v^2) + \varepsilon^2\{[v\omega/(1-v^2)]^2 - [\omega/(1+v)]^2\}$.

Substitution of Eq. (23) into the $x = 1$ boundary conditions yields

$$\begin{pmatrix} 1 & 1 \\ r_2^2 & r_4^2 \end{pmatrix} \begin{pmatrix} b_2 \\ b_4 \end{pmatrix} = -\begin{pmatrix} 1 \\ r_1^2 \end{pmatrix} b_1 \quad \Rightarrow \quad \begin{pmatrix} b_2 \\ b_4 \end{pmatrix} = \frac{1}{r_4^2 - r_2^2} \begin{pmatrix} r_1^2 - r_4^2 \\ r_2^2 - r_1^2 \end{pmatrix} b_1. \tag{24}$$

For the two propagating waves, the relative phase due to the reflection at the right boundary B is given by the phase of

$$R_B = \frac{b_2}{b_1} = \frac{r_1^2 - r_4^2}{r_4^2 - r_2^2}. \tag{25}$$

Similar to the handling of R_A, the phase angle of R_B is

$$\phi(R_B) = \pi + O(\varepsilon^3). \tag{26}$$

Substitution of Eqs. (15), (22), and (26) into Eq. (17) leads to

$$\frac{2\omega}{1 - v^2} - \frac{1}{2}\varepsilon^2\omega^3\left[\frac{1}{(1+v)^4} + \frac{1}{(1-v)^4}\right] + \cdots = 2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots \,. \tag{27}$$

This is an algebraic equation (for ω) with the small parameter ε multiplying the highest power. Application of the previously discussed algebraic perturbation technique leads to three different roots. Only the root from the straightforward expansion form is retained. The two roots from the singular expansion form are discarded because they yield complex roots, and physically the natural frequency ω must be real for subcritical speeds. Substitution of $\omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots$ into Eq. (27) leads to

$$\omega_n = n\pi[1 - v^2 + \varepsilon^2 n^2 \pi^2 (v^4 + 6v^2 + 1)/2 + \cdots], \quad n = 1, 2, 3, \ldots \,. \tag{28}$$

Fig. 3 compares the fundamental ($n = 1$) natural frequencies obtained from Eq. (28) with numerically exact solutions for different belt speed v and different bending stiffness. The approximation results are best for small bending stiffness and low axial belt speeds. This is because for such cases the four roots in Eq. (15) have the best perturbation approximation. For

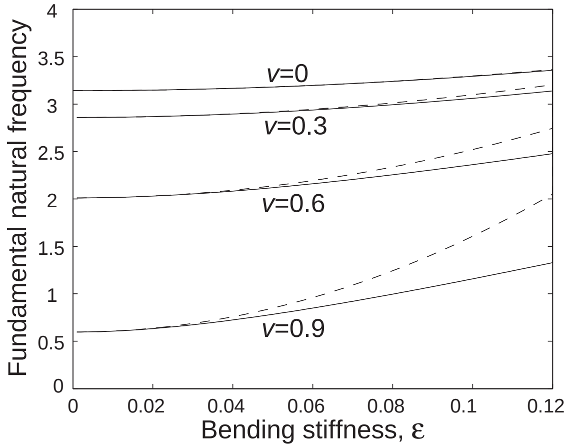

Fig. 3. Comparison of fundamental natural frequency for a simply supported beam: - - - , perturbation; —, exact.

large bending stiffness or high speed, more terms need to be incorporated in the perturbation approximation.

When $v = 0$, Eq. (28) becomes

$$\omega_n = n\pi\left[1 + \tfrac{1}{2}\varepsilon^2(n\pi)^2\right] + \cdots, \quad n = 1, 2, 3, \ldots\ . \tag{29}$$

The exact eigensolution for the special case $v = 0$ is

$$\omega_{exact} = \sqrt{(n\pi)^2 + \varepsilon^2(n\pi)^4}, \quad w(x) = \sin(n\pi x), \quad n = 1, 2, 3, \ldots\ . \tag{30}$$

Expansion of the eigenvalue in Eq. (30) for small ε yields Eq. (29).

Computation of the eigenfunctions requires additional consideration of the evanescent waves at the boundaries A and B. From Eqs. (20) and (24),

$$\tilde{R}_A = \frac{c_3}{c_2} = \frac{r_1^2 - r_2^2}{r_3^2 - r_1^2} = \frac{[\omega/(1+v)]^2 - [\omega/(1-v)]^2}{\{[v\omega/(1-v^2)]^2 - (1-v^2)/\varepsilon^2 - [\omega/(1+v)]^2\} + \mathrm{i}2v\omega/(\varepsilon\sqrt{1-v^2})}, \tag{31}$$

$$\tilde{R}_B = \frac{b_4}{b_1} = -\frac{r_1^2 - r_2^2}{r_4^2 - r_2^2} = -\frac{[\omega/(1+v)]^2 - [\omega/(1-v)]^2}{\{[v\omega/(1-v^2)]^2 - (1-v^2)/\varepsilon^2 - [\omega/(1-v)]^2\} - \mathrm{i}2v\omega/(\varepsilon\sqrt{1-v^2})}. \tag{32}$$

The eigenfunction can be written as

$$w(x) = c_1 \mathrm{e}^{\mathrm{i}r_1 x} + c_2 \mathrm{e}^{\mathrm{i}r_2 x} + c_3 \mathrm{e}^{\mathrm{i}r_3 x} + b_4 \mathrm{e}^{\mathrm{i}r_4 \xi}. \tag{33}$$

Normalization of Eq. (33) by dividing it by c_2, application of Eqs. (21), (31), (32), and use of $\xi = x - 1$ and $b_1 = c_1 \mathrm{e}^{\mathrm{i}r_1}$ give

$$w(x) = R_A \mathrm{e}^{\mathrm{i}r_1 x} + \mathrm{e}^{\mathrm{i}r_2 x} + \tilde{R}_A \mathrm{e}^{\mathrm{i}r_3 x} + R_A \mathrm{e}^{\mathrm{i}r_1} \tilde{R}_B \mathrm{e}^{\mathrm{i}r_4(x-1)}. \tag{34}$$

When $v \neq 0$, there are boundary layer terms from the evanescent r_3 and r_4 terms. But when $v = 0$, $R_A = -1$ and $\tilde{R}_A = \tilde{R}_B = 0$, leading to the eigenfunctions

$$w(x) = -\mathrm{e}^{\mathrm{i}r_1 x} + \mathrm{e}^{\mathrm{i}r_2 x}. \tag{35}$$

Substitution of Eqs. (15), (29), and $v = 0$ into Eq. (35) gives the eigenfunction approximation

$$w(x) = \sin(n\pi x), \quad n = 1, 2, 3 \ldots, \tag{36}$$

in agreement with the exact solution in Eq. (30). The eigenfunctions have no boundary layer terms for $v = 0$ while they do for $v \neq 0$.

4. Other boundary conditions

The above perturbation method for the eigenvalue problem can be applied to other boundary conditions. Table 2 lists the reflection coefficients $(R_A, R_B, \tilde{R}_A, \tilde{R}_B)$ for two different end supports. These can be used to determine the eigensolutions for combinations of such boundary conditions. For example, for the fixed-simple boundary conditions $w(0,t) = w_x(0,t) = 0$, $w(1,t) = w_{xx}(1,t) = 0$, we have $R_A = (r_2 - r_3)/(r_3 - r_1)$, $\tilde{R}_A = (r_1 - r_2)/(r_3 - r_1)$, $R_B = (r_1^2 - r_4^2)/(r_4^2 - r_2^2)$,

Table 2
Reflection coefficients for different end supports

Boundary type	A $(x=0)$	B $(x=1)$
Simple support	$R_A = -\dfrac{r_3^2 - r_2^2}{r_3^2 - r_1^2}$	$R_B = -\dfrac{r_4^2 - r_1^2}{r_4^2 - r_2^2}$
	$\tilde{R}_A = \dfrac{r_1^2 - r_2^2}{r_3^2 - r_1^2}$	$\tilde{R}_B = -\dfrac{r_1^2 - r_2^2}{r_4^2 - r_2^2}$
Fixed support	$R_A = -\dfrac{r_3 - r_2}{r_3 - r_1}$	$R_B = -\dfrac{r_4 - r_1}{r_4 - r_2}$
	$\tilde{R}_A = \dfrac{r_1 - r_2}{r_3 - r_1}$	$\tilde{R}_B = -\dfrac{r_1 - r_2}{r_4 - r_2}$

and $\tilde{R}_B = (r_2^2 - r_1^2)/(r_4^2 - r_2^2)$. The presented method yields the approximate eigenvalues

$$\omega_n = n\pi(1 - v^2)\left[1 + \frac{1}{\sqrt{1 - v^2}}\varepsilon + \cdots\right], \quad n = 1, 2, 3\ldots. \tag{37}$$

The corresponding eigenfunctions still have form (34) but with revised coefficients R_A, $\tilde{R}_A$, $\tilde{R}_B$. The roots r_1–r_4 in Eq. (15) have the same functional form but differ between boundary condition cases because of changes in the expression for ω (e.g. Eqs. (27) and (37)).

Another example is for the fixed-fixed boundary conditions $w(0,t) = w(1,t) = 0$, $w_x(0,t) = w_x(1,t) = 0$. The approximate eigenvalues are

$$\omega_n = n\pi(1 - v^2)\left[1 + \frac{2}{\sqrt{1 - v^2}}\varepsilon + \cdots\right], \quad n = 1, 2, 3\ldots\ . \tag{38}$$

The approximate eigenfunction is the superposition of the four component waves. Again, Eq. (34) holds for this case, in which ω is different for r_1–r_4 in Eq. (15) according to Eq. (38), and the coefficients R_A, $\tilde{R}_A$, and $\tilde{R}_B$ are available in the second row of Table 2.

For the case of fixed–fixed supports, letting $v = 0$ in Eq. (38) gives

$$\omega_n = n\pi(1 + 2\varepsilon) + \cdots, \quad n = 1, 2, 3\ldots \tag{39}$$

with corresponding eigenfunctions

$$w_n(x) = \frac{\sin(n\pi x)}{n\pi} + \varepsilon[-(1 - 2x)\cos(n\pi x) + \mathrm{e}^{-x/\varepsilon} + (-1)^{n+1}\mathrm{e}^{(x-1)/\varepsilon}] + \varepsilon^2(\ldots), \quad n = 1, 2, 3\ldots. \tag{40}$$

For this zero speed case, by using a different perturbation method (directly assuming the eigenfunction as the combination of outer solution and boundary layer inner solutions), O'Malley [16] obtains the approximate eigensolutions

$$\omega_n^2 = (n\pi)^2 + 4\varepsilon(n\pi)^2 + \cdots, \quad n = 1, 2, 3\ldots, \tag{41}$$

$$w_n(x) = \frac{\sin(n\pi x)}{n\pi} + \varepsilon\left[\frac{\sin(n\pi x)}{n\pi} - (1 - 2x)\cos(n\pi x) + \mathrm{e}^{-x/\varepsilon} + (-1)^n\mathrm{e}^{(x-1)/\varepsilon}\right] + \varepsilon^2(\ldots), \quad n = 1, 2, 3\ldots \tag{42}$$

These eigenvalues are the same as Eq. (39) for the leading terms. There are two differences between Eqs. (40) and (42). First, in Eq. (42), the coefficient before the boundary layer term at $x = 1$ is $(-1)^n$ while in Eq. (40) the coefficient is $(-1)^{n+1}$. Second, there is no term in Eq. (40) corresponding to the $\sin(n\pi x)/(n\pi)$ term inside the bracket of Eq. (42). For the first difference, it can be easily checked that $(-1)^n$ in Eq. (42) is a typographical error that should be $(-1)^{n+1}$. The second difference is due to different normalization used in Eqs. (42) and (40); multiplication of Eq. (40) by $(1 + \varepsilon)$ yields the same leading terms as Eq. (42). Thus, the eigenfunctions equations (40) and (42) agree with each other. For such boundary conditions, there are always boundary layer terms at the ends of the beam. O'Malley's method is only for stationary beams and does not extend simply to problems of axially moving beams.

5. Conclusion

Perturbation techniques for algebraic equations and the phase closure principle are combined to analyze the free vibration of axially moving beams with small bending stiffness. Closed-form approximate natural frequencies and vibration modes are derived based on the propagation and attenuation properties of the constituent waves. Uniformly valid approximate eigenfunctions are obtained. Different combinations of boundary conditions can be readily handled. Boundary layers, when present, are incorporated via evanescent waves. When the axial speed of the beam is zero, the solutions converge to known solutions for these non-gyroscopic systems. The approach is straightforward, suited for different boundary conditions, and has accessible physical explanation.

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